

Strong Bell violations in N level systems

V. Ravishankar[°] and Radha Pyari Sandhir[•]

[°]Department of Physics, Indian Institute of Technology Delhi, Hauz Khas, New Delhi, India 110016

[•] Department of Physics and Computer Science, Dayalbagh Educational Institute, Dayalbagh Rd, Dayalbagh, Agra, Uttar Pradesh, India 282005

E-mail: [°]v ravi@physics.iitd.ac.in, [•]radha.pyari@gmail.com

Abstract. We study the variety of ways in which nonlocality can be experimentally detected in N level systems, with conventional Bell inequality violation. By simple constructions, show that there are an exponentially large number of correlations that violate the inequality strongly, which remain robust at any N . We further formulate the concept of weak classical limit to get a better insight to the so called classical behaviour that quantum systems are expected to exhibit when $N \rightarrow \infty$.

PACS numbers: 03.65.Ud, 03.67.Mn, 03.65.Aa

Keywords: N level systems, Nonlocality, Bell inequality

1. Introduction

The focus of Bell's seminal paper on non-locality was a two qubit system [1] (see also the precursor work of Bohm [2]). It has since then been studied exhaustively, both experimentally [3, 4, 5, 6, 7, 8, 9, 10], and theoretically [11, 12, 13, 14, 15, 16, 17], to name a few. It appears that with the plugging of the two notorious loopholes – of detection and of locality [10] – it would be safe to conclude that the validity of quantum mechanics rests on a firm experimental pedestal.

Be that as it may, the above successes do not necessarily mean that one has exhausted a study of all aspects of quantum mechanics, or for that matter, all tests of nonlocality. Classical limit of quantum mechanics, which is closely coupled with the measurement problem, is one such example, and two qubit systems are not ideal candidates for such a study. It is also necessary to devise as many tests of nonlocality as possible, since one may envisage that some time in the near future, the focus will not be merely establishing nonlocality, but also on distinguishing between different nonlocal theories that are incompatible with the tenets of quantum mechanics [18, 19, 20, 21]. The interplay amongst various criteria for the quantumness, such as nonlocality, entanglement (separability), and coherence [22, 23], is yet another important issue if one wishes to understand novel models of classical computation [24] and generalized

local hidden variable theories [25]. Coupled N level systems are natural candidates to move in these directions.

The purpose of this paper is to explore the extent and the variety of correlations through which nonlocality can manifest in coupled N level systems. They could arise in a number of ways. One could harness N convenient levels of an infinite dimensional system, or couple multiple degrees such as spin with orbital angular momentum of a photon as recently accomplished [26], or create high orbital angular momentum states of photons [27, 28, 29]. Exploitation of spatial modes using fibres has also been proposed for preparing entangled N level systems [30, 31].

Apart from the issues raised above, entangled N level systems also serve a practical purpose. Information capacity of a channel in quantum key distribution is known to increase with the number of orthogonal states available [32, 28], and quantum key distribution becomes more secure as the dimension increases [33, 34, 35, 36, 37]. As pointed out in [29], orbital degree of freedom of photons offer “unlimited” quantum numbers which may constitute a scalable quantum resource [38, 21, 39, 40].

There are a number of studies of nonlocality in higher dimensions [41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60]. The strategies have been to devise *new* inequalities whose very structure may depend on N , and/or to consider one specific correlation to study the violation in a given N level system. Depending on the inequality employed, and the correlation studied, divergent conclusions have been drawn: that in the large N limit either the classical limit is reached [41, 42], or not reached [44], or that nonlocality becomes stronger as $N \rightarrow \infty$ [48]. On the other hand, there are also well known results that a system becomes ‘classical’ in the large N limit [61].

The purpose of this paper is to point out that these issues can be settled, to a large extent, by undertaking a comprehensive study of correlations for systems upto $N = 11$, and by combining them with other known results [44]. To accomplish this we adopt a strategy which differs from the earlier ones in two crucial aspects.

- (i) We observe that the original formulation of nonlocality [1] does not mandate that the bipartite state comprise of two qubits; it merely demands that the observables be bounded by unit norm, irrespective of the dimension of the system. For that reason, we adhere to the standard formulation of nonlocality, Eq. 2, for all dimensions. This has the added advantage that one can meaningfully compare results across different values of N .
- (ii) We do not restrict our attention to any particular correlation, but explore a large family of correlations, whose membership grows exponentially with the dimension of the state.

Our main results may be summarised thus: We find that at any given N , there are an exponentially large number of correlations that exhibit strong nonlocality. While some of the correlations cease to violate the inequality at a threshold value of N , new correlations arise that compensate for the reduction in the number. These conclusions

are based on our complete results upto $N = 11$, random simulations for $N = 13, 21$, and an early example worked out in [44].

2. The Setting

2.1. The State

Throughout the paper, we employ the language of spin for convenience; $N = 2s + 1$. Maximally entangled pure states exhibit maximal nonlocality which, in turn, is invariant under local operations. We choose, without loss of generality, the rotationally invariant spin singlet state,

$$|\Psi\rangle = \frac{1}{\sqrt{2s+1}} \sum_m (-1)^{s-m} |m, -m\rangle. \quad (1)$$

2.2. The Inequality

We employ the Bell inequality in one of its standard forms [1, 62], and define the Bell function:

$$\mathcal{B} \equiv |\mathcal{C}(a, b) - \mathcal{C}(a, b')| + |\mathcal{C}(a', b) + \mathcal{C}(a', b')|. \quad (2)$$

Non-locality is exhibited by $|\Psi\rangle$ if $\mathcal{B} > 2$ in some region in the parameter space. To identify the parameter space, we note that the isotropic state is invariant under the larger group of transformations, *viz*, $SU(N) \times SU(N)$. For simplicity, we restrict our observables and correlations to those that pertain to the smaller group of transformations, i.e., $SU(2) \times SU(2)$. This makes the visualization of correlations simpler, and an experimental verification easier. It also suffices to demonstrate most of the desired features. We briefly comment on the import that the larger set of transformations would have, in Sect. 6.1. With this simplification, the experimental set up that probes nonlocality may be depicted as shown in Fig. 1.

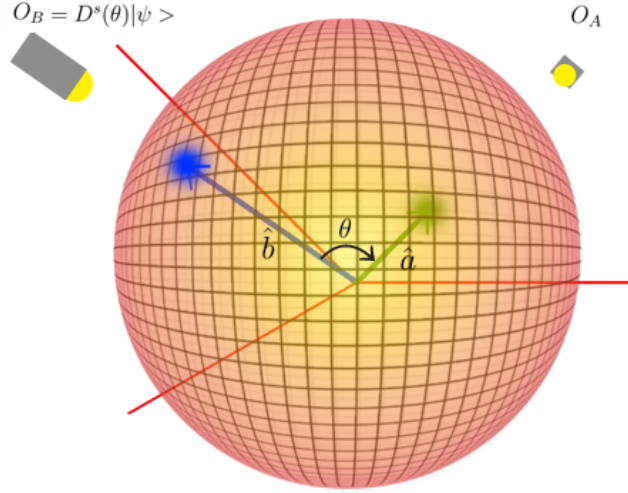


Figure 1. Detector configuration employed to measure observable O_A along quantisation axis \hat{a} and O_B along quantisation axis \hat{b} .

The observables of the two subsystems are measured along their respective quantization axes \hat{a} and \hat{b} .

2.3. The Observables

We now address the most important question, of the choice of observables for the two subsystems. This question is rather trivial for a two qubit system since there is, essentially, only one unique correlation. For a spin s state, there are $\mathcal{N} = N^2 - 1$ independent observables and the number of independent correlations is given by $\binom{\mathcal{N}}{2} \sim O(\exp(N^2))$. Needless to say, this enumeration is also not of much use because one has to still figure out the right combination of the observables, and their corresponding correlations. Many of them would not reveal the nonlocal features inherent in the state. Thus a well informed choice is required.

Motivated by the two qubit case, the example worked out in [44], and a desire to devise experimentally friendly correlations, we propose the family of observables for the subsystems to be of the form:

$$O(\hat{q}) = \sum (-1)^{f_m} |m\rangle \langle m| \leftrightarrow \mathbb{P} = \sum_{m=-s}^{m=+s} f_m 2^{m-s}; \quad (3)$$

Here \hat{q} refers to the choice of the quantization axis and f_m take values $\{0, 1\}$. The sequence $\{f_m\}$ - the parity bit array - characterizes the observable. In turn, each parity bit array admits a unique integer representation \mathbb{P} , which we may dub as the parity bit

integer. We employ this notation throughout. In short, we have shortlisted a set of 2^N observables for each subsystem. Note that the result of a measurement of $O(\hat{q})$ can be easily expressed in terms of appropriate count rates for various quantum numbers m .

2.4. The correlations

Since the subsystems are treated on an equal footing in the state, we choose identical observables for A and B , both labelled by the same \mathbb{P} , by virtue of which it also characterises the correlator $O_A O_B$. Thus we are equipped with a set of 2^N correlations of which the one proposed by Peres [44] is but one example.

The two observables differ only in their quantization axes, which we denote by \hat{a}, \hat{b} respectively. We denote the corresponding spin bases by $\{|m\rangle\}$ and $\{|n\rangle\}$. The two bases are related via the N dimensional irrep of $SU(2)$. Explicitly,

$$|n\rangle = \sum_m D_{m,n}^s(\phi, \theta, \psi) |m\rangle. \quad (4)$$

The correlation function, given by

$$C(\hat{a}, \hat{b}) = \langle \Psi | O_A(\hat{a}) O_B(\hat{b}) | \Psi \rangle, \quad (5)$$

is invariant under rotations, and depends only on $\cos \theta \equiv \hat{a} \cdot \hat{b}$. It is found to be

$$C(\hat{a}, \hat{b}) = \frac{1}{2s+1} \sum_{n,m} (-1)^{f_n+f_m} |d_{-m,n}^s(\theta)|^2 \quad (6)$$

where the d matrix represents a rotation about the y axis by an angle θ .

2.5. Detector Configuration

The detector configuration considered throughout the paper is the standard planar configuration which corresponds to the geometry

$$\theta_{ab} = \theta_{a'b} = \theta_{a'b'} = \frac{\theta_{ab'}}{3}, \quad (7)$$

This prescription is not *ad hoc* for two reasons. An extensive study, partly analytical and partly numerical has shown this geometry to be optimal [63]. Our own computation confirms that maximal violations occur at this geometry. The parameter space is thus rendered one dimensional.

3. Computational Complexity

We have at hand a set of 2^N correlations, of which the one corresponding to $f_m = 1$ for all m is trivial. Furthermore, an overall parity operation on any observable leaves the Bell function intact, leading to $2^{N-1} - 1$ independent correlations. It is possible that two physically distinct correlations $\mathbb{P}_i, \mathbb{P}_j$ may yield the same expectation value $C(\hat{a}, \hat{b})$. Before we identify distinct correlation functions, we quickly examine the computational issues involved in evaluating the elements of the irrep $SU(2)$, the Wigner- D matrices.

3.1. The Structure of the Wigner-D Matrix

The elements of the D matrix, occurring in Eq. 6, do admit an analytic expression for any spin, but their form becomes increasingly complicated to evaluate with increasing dimensions. Probing nonlocality analytically also becomes increasingly frustrating, barring some exceptional cases [44]. We employ quasi numerical techniques to evaluate the correlation as well as the Bell function.

Table 1. Number of Distinct Correlations for Spin- s .

s	$D(\mathcal{H})$	N_D	\mathcal{V}_C^s	\mathcal{N}_C^s	s	$D(\mathcal{H})$	N_D	\mathcal{V}_C^s	\mathcal{N}_C^s
1	9	2	3	2	3/2	16	4	7	5
2	25	6	15	9	5/2	36	9	31	19
3	49	12	63	35	7/2	64	16	127	71
4	81	20	255	135	9/2	100	25	511	271
5	121	30	1023	527	11/2	144	36	2047	1055
6	169	42	4095	2079	13/2	196	49	8191	4159
7	225	56	16383	8255					

s is the spin value, $D(\mathcal{H})$ is the dimension of the Hilbert space, N_D is number of unique d -matrix elements, \mathcal{V}_C^s is number of distinct parity bit integers, and \mathcal{N}_C is number of distinct correlations.

Being irreps of $SU(2)$, the Wigner matrices respect the following symmetries:

$$|d_{m_1 m_2}|^2 = |d_{-m_1 - m_2}|^2 = |d_{m_2 m_1}|^2 = |d_{-m_2 - m_1}|^2 \quad (8)$$

which essentially obviates the need to evaluate 75% of the matrix elements in the matrix. The number of independent matrix elements is given by $N_D = s(s+1)$ for integer s and $N_D = (s + \frac{1}{2})^2$ for half odd integer s . Also, the sum of the diagonal elements is common to all the correlations. This mitigates the computational difficulty marginally, and one is still left with the task of executing a sum over terms of $O(N!)$ for each matrix element.

The symmetries also allow different correlations to yield the same expectation value in Eq. 6. Table 1 illustrates the points made, where we display the details upto $s = 7$ ($N = 15$). Of particular interest are the last two columns which enumerate the total number of correlations \mathcal{V}_C^s , and the number of distinct correlation functions, \mathcal{N}_C^s . One may surmise that $\mathcal{V}_C^s \sim \frac{1}{2}\mathcal{N}_C^s$ which still leaves us with an exponentially large number to reckon with. The results on these distinct correlations will be presented systematically in the following sections.

4. Results

We present results for seven values of spin, $s = 1, \dots, 5/2$ and $s = 3, 4, 5$. Overall, our analysis covers 1397 independent correlations, of which 732 have distinct functional

forms. We also present results for $s = 6, 10$ for two randomly chosen correlations each.

4.1. $s=1$

For this simplest case, there are three independent correlations, with two of them having identical correlation functions. The relevant results are summarized in Table 2 which exhibits the observable (the parity bit integer \mathbb{P}), the algebraic expressions for the correlation functions, and the maximum violation predicted. We see that all the correlations yield large violations. The Peres correlation [44] corresponds to the special case $\mathbb{P} = 5$.

Table 2. Bell Violations for Spin-1.

No.	\mathbb{P}	$\mathcal{C}(\theta)$	\mathcal{B}_{\max}
1	6, 4	$(\cos^2 \theta + \cos \theta)/3$	2.48
2	5	$(4 \cos^2 \theta - 1)/3$	2.55

\mathbb{P} is the parity bit integer, $\mathcal{C}(\theta)$ is the algebraic expression of the correlation function, and \mathcal{B}_{\max} is the maximum Bell violation.

4.2. Spin- $\frac{3}{2}$ Case.

This case presents a richer landscape, with a total of seven independent correlations and five distinct correlation functions. The algebraic expressions for the corresponding correlations, and the extent of violations, are collected in Table 3. Note that the maximum violation occurs for $\mathbb{P} = 9$, for which the Bell function attains a value 2.62. It is, again, important to observe that a good violation is seen in all the five correlations.

Table 3. Bell Violations for Spin- $\frac{3}{2}$.

No.	\mathbb{P}	$\mathcal{C}(\theta)$	\mathcal{B}_{\max}
1	14,8	$(\cos^3 \theta - \cos^2 \theta + 3 \cos \theta + 1)/8$	2.348
2	13,11	$(3 \cos^3 \theta + 3 \cos^2 \theta - 5 \cos \theta + 1)/8$	2.401
3	12	$(\cos^3 \theta + \cos \theta)2$	2.349
4	10	$2 \cos^3 \theta - \cos \theta$	2.51
5	9	$(\cos^2 \theta - 2 \cos \theta - 1)/2$	2.62

\mathbb{P} is the parity bit integer, $\mathcal{C}(\theta)$ is the algebraic expression of the correlation function, \mathcal{B}_{\max} is the maximal Bell violation.

5. Bell Violations for $s \geq 2$

Since the number of correlations grows exponentially with N , it is expedient to employ a different way of presenting the results, especially because we find that every single correlation is found to lead to a large violation. We present our results for each spin by

plotting the number of correlations against the value of the Bell function, rather than display the results for each of them. Figs. 2 and 3 show histograms for the number of distinct correlations that show a particular violation, for seven different values of s .

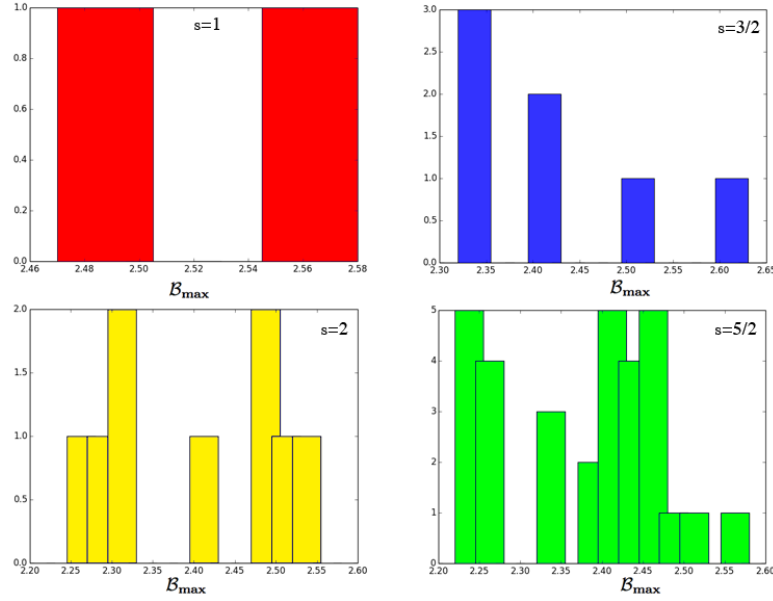


Figure 2. Bell function maxima, \mathcal{B}_{\max} , for all *distinct* correlation functions. Red: $s = 1$, Blue: $s = \frac{3}{2}$, Yellow: $s = 2$, Green: $s = \frac{5}{2}$, Purple: $s = 3$, Orange: $s = 4$, Teal: $s = 5$. Note: all correlations violated the Bell inequality, i.e., there were no correlations for which $\mathcal{B}_{\max} < 2$.

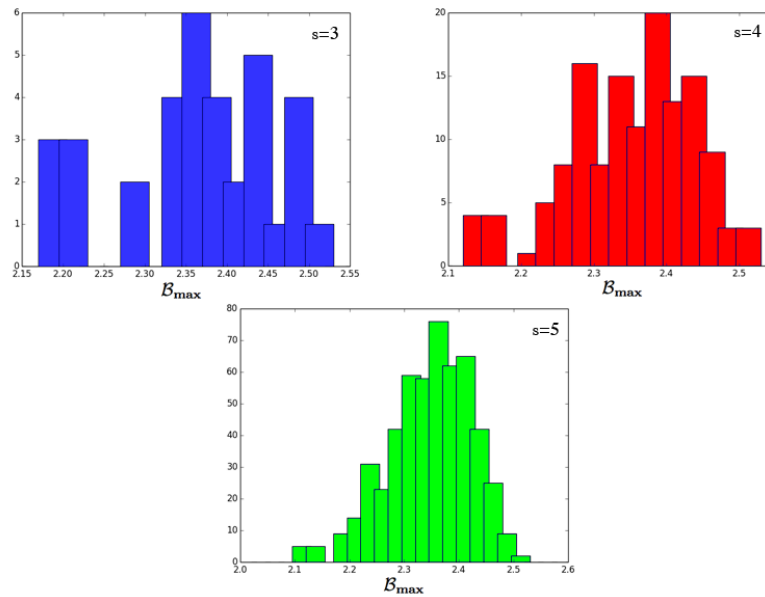


Figure 3. Bell function maxima, \mathcal{B}_{\max} , for all *distinct* correlation functions. Blue: $s = 3$, Red: $s = 4$, Green: $s = 5$. Note: all correlations violate the Bell inequality, i.e., there were no correlations for which $\mathcal{B}_{\max} < 2$.

The correlations that show maximum violation are listed in Table 4.

Table 4. Maximum Bell Violations for Spin- s across all correlations.

s	$D(\mathcal{H})$	\mathbb{P}	\mathcal{B}_{\max}
1	9	5	2.55
$\frac{3}{2}$	16	9	2.62
2	25	17	2.53
$\frac{5}{2}$	36	54	2.56
3	49	77	2.51
4	81	306	2.51
5	121	1212	2.51

$D(\mathcal{H})$ is the dimension of the Hilbert space, \mathbb{P} is the parity bit vector, and \mathcal{B}_{\max} is the maximum violation across all correlations.

As mentioned, it gets increasingly time intensive to compute the Bell functions for $s > 5$. We have verified violation in dimensions $N = 6, 10$ for a few correlations, as shown in Table 5.

Table 5. Bell Violations for Select Correlations of Spin- s .

s	$D(\mathcal{H})$	\mathbb{P}	\mathcal{B}_{\max}
6	169	4097	2.20
6	169	5461	2.47
10	441	524289	2.12
10	441	1398101	2.47

$D(\mathcal{H})$ is the dimension of the Hilbert space, and \mathcal{B}_{\max} is the maximum violation for the correlation labelled by \mathbb{P} .

6. Discussion of Results

The results show that there is credible and strong evidence that nonlocality thrives and is undiminished in higher dimensional systems. We have also constructed an exponentially large number of correlations that exhibit nonlocality, thereby providing a fertile ground for experimentalists to test nonlocality. They also serve to quantify, in a reliable manner, the utility of quNits for quantum information processes such as teleportation or quantum key distribution. What more, the correlations require an adaptation of the standard Stern Gerlach set-ups. Since the definition of the Bell function is independent of N , violations are easily comparable across various dimensions.

The study also raises questions regarding the large N limit, and the magnitude and range of violations, which we address in the following subsections.

6.1. Magnitude of Violation

We first comment on the relative magnitudes of violation seen in the previous section, before we move on to address more general issues. The trivial choice corresponding to $f_m = 1$, the identity operator, yields no violation. Minor deviations from identity may be expected to yield relatively small violations compared to those with larger ones. This is vindicated by the results shown in Tables 2 and 3. The magnitude of the violation is larger if the overlap of the correlator with identity operator is smaller. In fact, it is very well possible that those small perturbations may even stop yielding violations beyond a certain N . To check that, consider the observable $O_A = I - 2|-s\rangle\langle -s|$ corresponding to the choice $f_m = 1 - \delta_{m,-s}$. This has the maximum overlap with the identity, with $\text{Tr}(IO_A) = 2s - 1$. The corresponding parity bit numbers are $\mathbb{P} = 6, 14$, respectively for $s = 1, 3/2$, both of which show minimum violation as seen in Tables 2 and 3. We have verified that the violation becomes weaker with increasing N , and the violation is definitely not seen when $N = 50$. This does not, however, dent the robustness of the results since every increment in the dimension leads to an exponential increase in the number of observables.

The maximum possible violation has been the most engaging question in studies of nonlocality. The speculation that the violation dies off as $N \rightarrow \infty$ has been negated by this study, as well as in [42, 45, 44, 48]. Two questions persist, however. The Cirel'son bound, $\mathcal{B} = 2\sqrt{2}$, is not attained by any of the correlations, though such correlations are guaranteed to exist [64]. Secondly, it is plausible that the range of angles over which the violation occurs vanishes as $N \rightarrow \infty$, as in [44].

We first note that the list of correlations that we have studied is not exhaustive. Any N level system has pseudo M level subsystems, $M = 2, 3, \dots, N - 1$. Hence the violations reported for lower dimensions are inherited by the higher dimension as well. It would be incorrect to argue that they do *not* represent genuine nonlocality, for, one looks at – not a particular M level system, but the set of $\binom{N}{M}$ systems, for each constituent. The family of correlations in all these subsystems span the parent state, and reflect the genuinely quantum nature of the state. Else one would have to draw the paradoxical conclusion that the parent state is less quantum – in spite of being pure – than the projected states.

Does this mean that a correlation that genuinely spans the full space is bound to give a lesser violation? To answer this question, we note that the family of correlations considered here, and elsewhere, is a small subset of all possible correlations. Given two orthonormal bases $\{|m_i\rangle\}$ and $\{|n_i\rangle\}$, the most general transformation that connects them is a member of $U(N)$, rather than irreps of $SU(2)$. The parameter space is, therefore, of higher dimensions, and that may be expected to yield the maximum violation without diminution in range. This is a topic of separate investigation, and will be reported elsewhere.

6.2. Large N as Weak Classical Limit

The behaviour of quantum systems in the large N holds an abiding interest, in view of the “proofs” that a system becomes classical in that limit [61, 65]. These proofs exploit an essential property of spin coherent states: though they are over complete in any dimension, they become increasingly orthogonal as $N \rightarrow \infty$. Delicate that this limit is, it needs to be tested independently *vis-a-vis* nonlocality, as has been done in [41, 44, 42, 45, 48] – with some of them purporting to demonstrate the classical behaviour, and the others – to the contrary. The findings of this paper demonstrate that for any N , there are many correlations that show a strong signal for nonlocality. A reconciliation between the two results is possible through the concept of weak classicality.

Suppose that one goes from a dimension $N - 1$ to N . This introduces new independent observables which are, to be precise, $2N - 1$ in number. They are, in our case, the irreducible tensor operators,

$$T_q^{2s}(\vec{S}) \equiv C_{2s}(\vec{S} \cdot \vec{\nabla})^{2s} Y_{2s,q}(\hat{n}); \quad q = -2s, \dots, 2s. \quad (9)$$

These observables *do not* have counterparts in lower dimensions. In the language of measurements, and of statistics, they are the moments of the spin operator of degree $2s$, and constitute new experimental resources that emerge when the dimension of the Hilbert space is increased by unity.

Consider, now, the overlap of the observables defined in Eq. 3, with T_0^{2s} ,

$$\begin{aligned} \text{Tr}\{OT_0^{2s}(\vec{S})\} &= \sum_m (-1)^{f_m} \langle s, m | T_0^{2s}(\vec{S}) | s, m \rangle \\ &= \left\{ \sum_m (-1)^{f_m} C(s \ 2s \ s; s \ 0 \ s) \right\} \langle s || T^{2s} || s \rangle. \end{aligned} \quad (10)$$

It is generically nonvanishing, except when f_m takes a constant value, in which case, the observable would be the trivial Identity operator. The matrix element of these operators are, therefore, strongly dependent on N , however large N may be.

The above observation leads us naturally to the notion of weak classical limit. In other words, there is a demand for experimental techniques to determine the new observables. Suppose that one has resources to measure tensor moments upto some maximum order k_{max} . Then the determination of our correlations beyond a threshold value of spin would be impossible, or grossly imprecise. And since a pure state is largely characterised by the highest order tensor moment [66, 67], the lower rank observables are not expected to reveal nonlocality. We have, in fact, verified that that this does happen: nonlocality fails to show up if $s \gtrsim k_{max}$ [68]. We call this the weak classical limit.

In short, even with ideal measurements, quantumness may not be revealed if the experimental resources are limited in the sense described above. This conclusion complements other results on classical limits which invoke decoherence [69, 70, 71] or coarse grained measurements [72], and of the experimental resolution of adjacent states in a Stern Gerlach set up [44]. In all these cases, classical limit is not seen as an inherent

property of the system, but as arising out of extraneous exigencies, either because of environment, or precision, or lack of resources. Any of them may mask the quantumness of the state.

7. Conclusion

We have shown that coupled N level systems exhibit nonlocality through large violations of the Bell inequality by an exponentially large number of correlations. We have also introduced the idea of weak classical limit which allows us to understand the large N limit of quantum systems. The study also sets course for future investigations where the bases that define the states of the subsystems are related by the most general unitary transformations. An extension of these results to multipartite quNit systems also holds promise to study the role of environment in greater detail.

Acknowledgements

We are grateful for our insightful interactions with Konrad Banaszek. RPS thanks the Department of Science and Technology (DST), India for enabling funding her research under the WOS-A Women's Scientist Scheme.

- [1] Bell J S 1964 *Physics* **1** 195
- [2] Bohm D and Aharonov Y 1957 *Phys. Rev.* **108**(4) 1070
- [3] Aspect A, Grangier P and Roger G 1982 *Phys. Rev. Lett.* **49**(2) 91
- [4] Ou Z Y and Mandel L 1988 *Phys. Rev. Lett.* **61**(1) 50
- [5] Ou Z Y, Pereira S F, Kimble H J and Peng K C 1992 *Phys. Rev. Lett.* **68**(25) 3663
- [6] Tapster P R, Rarity J G and Owens P C M 1994 *Phys. Rev. Lett.* **73**(14) 1923
- [7] Tittel W, Brendel J, Zbinden H and Gisin N 1998 *Phys. Rev. Lett.* **81**(17) 3563
- [8] Weihs G, Jennewein T, Simon C, Weinfurter H and Zeilinger A 1998 *Phys. Rev. Lett.* **81**(23) 5039
- [9] Rowe M A, Kielpinski D, Meyer V, Sackett C A, Itano M, Monroe C and Wineland D J 2001 *Nature* **409** 791
- [10] Hensen *et al.* 2015 *Nature* **526** 682
- [11] Clauser J F, Horne M A, Shimony A and Holt R A October 1969 *Phys. Rev. Lett.* **23** 880
- [12] Bell J S 1971 *Foundations of Quantum Mechanics* 171
- [13] Clauser J F and Shimony A 1978 *Reports on Progress in Physics* **41** 1881
- [14] Garg A and Mermin N 1984 *Foundations of Physics* **14** 1
- [15] Hill S and Wootters W K 1997 *Phys. Rev. Lett.* **78**(26) 5022
- [16] Wootters W K 1998 *Phys. Rev. Lett.* **80**(10) 2245
- [17] Banaszek K and Wódkiewicz K 1998 *Phys. Rev. A* **58**(6) 4345
- [18] Nelson E 1966 *Phys. Rev.* **150**(4) 1079
- [19] Davidson M P 1979 *Letters in Mathematical Physics* **3** 271
- [20] Leggett A 2003 *Foundations of Physics* **33** 1469
- [21] Gröblacher S, Paterek T, Kaltenbaek R, Brukner C, Zukowski M, Aspelmeyer M and Zeilinger A 2007 *Nature* **446** 871
- [22] Werner R F 1989 *Physical Review A* **40** 4277
- [23] Braunstein S, Mann A and Revzen M 1992 *Phys. Rev. Lett.* **68**(22) 3259
- [24] Valiant L G 2001 *Proceedings of the thirty-third annual ACM symposium on Theory of computing STOC '01* (ACM) p 114
- [25] Bharath H M and Ravishankar V 2014 *Phys. Rev. A* **89**(6) 062110
- [26] Wang X *et al.* 2015 *Nature Letters* **518** 516

- [27] Dada A *et al.* 2011 *Nature Physics* **7** 677
- [28] Romero J, Giovannini D, Franke-Arnold S, Barnett S M and Padgett M J 2012 *Proc. SPIE 8542, Electro-Optical Remote Sensing, Photonic Technologies, and Applications* **VI** 012334
- [29] Giovannini D, Romero J, Leach J, Dudley A, Forbes A and Padgett M J 2013 *Phys. Rev. Lett.* **110**(14) 143601
- [30] Bharadwaj D, Thyagarajan K, Karpinski M and Banaszek K 2015 *Phys. Rev. A* **91** 033824
- [31] Bharadwaj D, Thyagarajan K and Banaszek K 2015 *Frontiers in Optics 2015* (Optical Society of America) p FTu5A.3
- [32] Dixon P B, Howland G A, Schneeloch J and Howell J C 2012 *Phys. Rev. Lett.* **108**(14) 143603
- [33] Bechmann-Pasquinucci H and Tittel W 2000 *Phys. Rev. A* **61**(6) 062308
- [34] Bechmann-Pasquinucci H and Peres A 2000 *Phys. Rev. Lett.* **85**(15) 3313
- [35] Durt T, Kaszlikowski D, Chen J and Kwek L C 2004 *Phys. Rev. A* **69**(3) 032313
- [36] Walborn S P, Lemelle D S, Almeida M P and Ribeiro P H S 2006 *Phys. Rev. Lett.* **96**(9) 090501
- [37] Gruneisen M T, Black J P, Dymale R C and Stoltenberg K E 2012 *Proc. SPIE 8542, Electro-Optical Remote Sensing, Photonic Technologies, and Applications* **VI**, 85421Q
- [38] Mair A, Vaziri A, Weihs G and Zeilinger A 2001 *Nature* **412** 313
- [39] Leach J *et al.* 2010 *Science* **6** 662
- [40] Salakhutdinov V D, Eliel E R and Löffler W 2012 *Phys. Rev. Lett.* **108**(17) 173604
- [41] Mermin N D 1980 *Phys. Rev. D* **22** 356
- [42] Ardehali M 1991 *Phys. Rev. D* **44**(10) 3336
- [43] Popescu S and Rohrlich D 1992 *Physics Letters A* **169** 411
- [44] Peres A 1992 *Phys. Rev. A* **46**(7) 4413
- [45] Ardehali M 1992 *Phys. Rev. A* **46**(9) 5375
- [46] Gisin N and Peres A 1992 *Phys. Lett. A* **162** 15
- [47] Peres A 1999 *Found.Phys.* **29** 589 (*Preprint quant-ph/9807017*)
- [48] Kaszlikowski D, Gnaniński P, Żukowski M, Miklaszewski W and Zeilinger A 2000 *Phys. Rev. Lett.* **85**(21) 4418
- [49] Wu X, Zong H, Pang H and Wang F 2001 *Physics Letters A* **281** 203
- [50] Howell J, Lamas-Linares A and Bouwmeester D 2002 *Phys. Rev. Lett.* **88** 1
- [51] Collins D, Gisin N, Linden N, Massar S and Popescu S 2002 *Phys. Rev. Lett.* **88**(4) 040404
- [52] Collins D, Gisin N, Popescu S, Roberts D and Scarani V 2002 *Phys. Rev. Lett.* **88**(17) 170405
- [53] Moehring D L, Madsen M J, Blinov B B and Monroe C 2004 *Phys. Rev. Lett.* **93**(9) 090410
- [54] Fu L 2004 *Phys. Rev. Lett.* **92**(13) 130404
- [55] Son W, Lee J and Kim M 2006 *Phys. Rev. Lett.* **96**(6) 060406
- [56] Cai J, Zhou Z and Guo G 2007 *Quantum Info. Comput.* **7** 766
- [57] Gerry C C, Benmoussa A, Hach E E and Albert J 2009 *Phys. Rev. A* **79**(2) 022111
- [58] Lim J, Ryu J, Yoo S, Lee C, Bang J and Lee J 2010 *New Journal of Physics* **12** 103012
- [59] Gerhardt I, Liu Q, Lamas-Linares A, Skaar J, Scarani V, Makarov V and Kurtsiefer C 2011 *Phys. Rev. Lett.* **107**(17) 170404
- [60] Vermeyden L, Bonsma M, Noel C, Donohue J M, Wolfe E and Resch K J 2013 *Phys. Rev. A* **87**(3) 032105
- [61] Yaffe L G 1982 *Reviews of Modern Physics* **54**(2) 407
- [62] Bell J S 2004 *Speakable and Unspeakable in Quantum Mechanics: Collected Papers on Quantum Philosophy* (Cambridge University Press)
- [63] Mendaš I P 2005 *Phys. Rev. A* **71**(3) 034103
- [64] Landau L J 1987 *Physics Letters A* **120**(2) 54
- [65] Ghirardi G C, Rimini A and Weber T 1986 *Phys. Rev. D* **34**(2) 470
- [66] Ravishankar V and Ramachandran G 1986 *Modern Physics Letters A* **01** 333
- [67] Ravishankar V and Ramachandran G 1987 *Phys. Rev. C* **35**(1) 62
- [68] Ravishankar V and Sandhir R P 2015 *arXiv preprint quant-ph/1509.01805*
- [69] Zurek W 1991 *Physics Today* 36

- [70] Brune M, Hagley J, Dreyer X M, Maali A, Wunderlich C, Raimond J M and Haroche S 1996 *Phys. Rev. Lett.* **77**(24) 4887
- [71] Zurek W 2003 *arXiv preprint quant-ph/0306072*
- [72] Kofler J and Brukner C 2007 *Phys. Rev. Lett.* **99**(18) 180403